PHYSICAL REVIEW E, VOLUME 64, 026206

Phase synchronization of diffusively coupled Rössler oscillators with funnel attractors

H. L. Yang
Max-Planck-Institut für Physik Komplexer Systeme, Nöthnitzer Strasse 38, D-01187 Dresden, Germany
(Received 20 March 2001; published 17 July 2001)

Recently, an *antiphase* phase-synchronized state in a system of diffusively coupled Rössler oscillators has been reported [Gang Hu *et al.*, Phys. Rev. Lett. **85**, 3377 (2000)]. In the current paper this antiphase state is explored in detail. Our interests are concentrated on the comparison with the normal *in-phase* phase-synchronized state for phase-coherent oscillators and the effect of the lattice size. Our main results are that (i) this antiphase synchronization is only for funnel Rössler attractors and cannot be observed in a system of coupled phase-coherent oscillators; (ii) it can be observed only for intermediate values of the lattice size while it disappears for quite low or large values of the lattice size; and (iii) it is different from the in-phase phase-synchronized state of phase-coherent oscillators in many respects.

DOI: 10.1103/PhysRevE.64.026206 PACS number(s): 05.45.−a

I. INTRODUCTION

Chaos means that two trajectories starting from slightly different initial conditions will separate exponentially as time goes on [1]. A fascinating recent finding is that trajectories of such wild chaotic systems can be synchronized if they are properly coupled together [2]. Initially, the main interest of almost all researchers was focused on the case of coupled identical chaotic units [2–4]. Interesting phenomena such as on-off intermittency [3] and riddled basins [4] are found near the transition to the synchronized state. Most recently, the case of coupled nonidentical units [5–8] has attracted the attention of researchers due to the fact that parameter mismatches and stochastic perturbations are inevitable in real physical experiments and technical applications. Among such work, Rosenblum, Pikovsky, and Kurths showed the effect of phase synchronization of weakly coupled self-sustained chaotic oscillators [7–9]. They generalized the classic notion of phase locking for periodic oscillators and defined phase synchronization for autonomous chaotic oscillators as the appearance of certain relations between the phases of interacting systems. This phenomenon has been extensively studied both theoretically and experimentally [9].

Moreover [10], Hu *et al.* reported a study of a chain of diffusively coupled Rössler oscillators organized on a ring structure. The phenomenon of *antiphase* phase synchronization of chaotic oscillators was found. It is different from the previously reported results [11] where only *in-phase* phase synchronization was found for chaotic oscillators. In the current paper, this antiphase phase synchronization for Rössler oscillators is studied in detail and the results obtained are presented following this structure. In Sec. II, the model of coupled Rössler oscillators studied is presented. The in-phase state for phase-coherent attractors and the antiphase state for funnel attractors are shown. In Sec. III, the bifurcation to the antiphase state is characterized using both local and global variables. The relation with the Lyapunov exponent spectrum is also presented. A rough phase diagram is sketched. In Sec. IV, the effects of the lattice size and the inhomogeneity in natural frequencies are studied. Finally, the paper is concluded with a short discussion.

II. BASIC MODEL

A. The ring of oscillators

The model that we are interested in is a chain of diffusively coupled Rössler oscillators on a ring structure. It can be written as a set of ordinary differential equations,

$$\begin{aligned} \dot{x}_i &= -\omega_i y_i - z_i + \epsilon(x_{i+1} + x_{i-1} - 2x_i), \\ \dot{y}_i &= \omega_i x_i + a y_i + \epsilon(y_{i+1} + y_{i-1} - 2y_i), \\ \dot{z}_i &= 0.4 + (x_i - 8.5) z_i + \epsilon(z_{i+1} + z_{i-1} - 2z_i), \end{aligned} \tag{1}$$

with $x_{i+N}=x_i$, $y_{i+N}=y_i$, $z_{i+N}=z_i$. Here $i=1,\ldots,N$ is the index of the lattice site, ω_i is the natural frequency of an individual oscillator, and ϵ represents the coupling strength. In the following, the lattice size is fixed as $N=6$ and the natural frequency is set to $\omega_i=1$ for $i=1,\ldots,N$ except where other values are explicitly mentioned.

The parameter setting used here is different from the one in Ref. [10]. We prefer this parameter setting because it is popularly used in the literature especially in the study of the phase synchronization of Rössler oscillators [7–9,11]. Therefore, one can easily compare the present work with previous work on phase synchronization and see the differences and connection among them.

B. Phase-coherent attractor and funnel attractor

The parameter a determines the topology of the Rössler attractor. In changing the value of a, two kinds of attractor, namely, phase-coherent and funnel attractors, can be observed (Fig. 1). The critical value for the transition between them is found to be $a_c \simeq 0.21$. For $a=0.15<a_c$, the attractor is oriented so that its projection on the plane (x,y) exhibits a phase flow circulating around the origin. This is the phase-coherent attractor. Its phase can be simply defined as

$$\phi = \arctan\frac{y}{x}. \tag{2}$$

Two such oscillators are said to be phase synchronized if their phases satisfy the condition $\lim_{t\to\infty}|\phi_i(t)-\phi_j(t)| < \text{const}$.

 ©2001 The American Physical Society

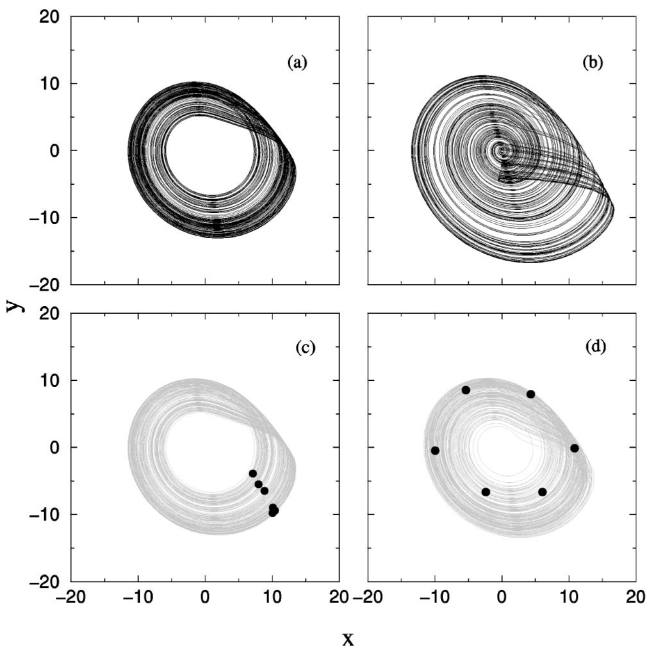

FIG. 1. (a) Phase-coherent attractor with $a=0.15$ and (b) funnel attractor with $a=0.25$ in a single Rössler oscillator without coupling. (c) In-phase phase synchronized state for coupled phase-coherent oscillators with $a=0.15$ and (d) antiphase phase synchronized state for coupled funnel oscillators with $a=0.25$. Here positions of oscillators are denoted by filled black circles and the attractor for the oscillator on site $i=1$ is shown in gray.

When the parameter a is larger than $a_c=0.21$, the topology of the Rössler attractor becomes quite complex and the phase flow is no longer oriented. The phase for this funnel attractor cannot be simply defined as for a phase-coherent attractor, but it is well known that for any autonomous system there exists a marginally stable variable corresponding to the zero-value Lyapunov exponent. In general this variable can be viewed as the phase for the attractor of this autonomous system although sometimes its value is difficult to determine; for instance, for the funnel Rössler oscillator. Recently a method for calculating the value of this phase variable has been reported that does not depend on the topology of the attractor [12].

C. In-phase and antiphase states

The in-phase state for a chain of phase-coherent Rössler oscillators is shown in Fig. 1(c) where all oscillators start from random initial conditions. After a transient period, all of them approach a nearly identical phase value, i.e., with $|\phi_i-\phi_j|\sim 0$. The snapshot distribution of oscillators is highly localized in the phase space. As time increases, this localized cloud of phase points rotates around the origin. In the plot, positions of oscillators are denoted by filled circles while the attractor for the oscillator on site $i=1$ is shown in gray. The coupling strength used is $\epsilon=0.005$. In comparing to an uncoupled oscillator, no obvious change in the attractor for the oscillator on site $i=1$ can be seen.

For $a=0.25$, the chain of oscillators has an antiphase state for the coupling strength ϵ in the interval $[0.045,0.093]$. The case shown in Fig. 1(d) is with $\epsilon=0.05$. Starting from random initial conditions and following transients, the oscillators self-organize into a state with a preferred phase lag between neighbor oscillators, i.e., with $\phi_{i+1}-\phi_i\simeq 2\pi/N$. Filled circles that denote oscillators' positions span the whole attractor. As time increases, this self-organized structure rotates around the origin while the phase lags among neighbor oscillators are kept nearly constant. In the plot, the attractor for the oscillator on site $i=1$ is shown in gray. Under coupling, it behaves like a phase-coherent oscillator and the trajectory avoids visiting the neighborhood of the origin.

III. BIFURCATION TO THE ANTIPHASE STATE

To characterize the change in the state of coupled oscillators here, we define a quantity

$$s(t)=\left\{\frac{1}{N}\sum_{i=1}^{N}\left[(x_i-\overline{x})^2+(y_i-\overline{y})^2+(z_i-\overline{z})^2\right]\right\}^{1/2} \tag{3}$$

with $\overline{x}=(1/N)\Sigma_{i=1}^{N}x_i$, and so on. As a state of the coupled oscillators is represented by a phase point in the $3N$-dimensional phase space, $s(t)$ can be viewed as the distance from this point to the diagonal $x_i=x_j$, $y_i=y_j$, and $z_i=z_j$ for any i and j. The average value $\langle s\rangle$ and the standard deviation $\sigma(s)$ over a long trajectory will be used to distinguish the in-phase and antiphase states. For an in-phase synchronized state, the difference between the states of oscillators is relatively small and the average distance $\langle s\rangle$ is expected to be small also. However, in the case of an antiphase synchronized state, this average distance $\langle s\rangle$ becomes quite large due to the almost constant phase lag. For a completely desynchronized state, oscillators can occasionally aggregate together, which induces a relatively small value of $s(t)$. They also have the chance to span the whole phase space, which leads to a large $s(t)$. This intermittency causes $\langle s\rangle$, for the desynchronized state, to take an intermediate value between those of the antiphase and in-phase states. The standard deviation $\sigma(s)$ is expected to be large here.

In Fig. 2, the evolution of $\langle s\rangle$ and $\sigma(s)$ with the coupling strength ϵ and the bifurcation diagram for the variable x_0 are plotted. In order to trace the bistable branches, for every coupling the end state of the last coupling is used as the starting state for the current coupling. Results for increasing and decreasing ϵ are shown in the same figure. Weak noise is used to destroy the unstable state. A large hysteresis loop is found for $0.057<\epsilon<0.093$. This was studied thoroughly in Ref. [10]. Here, we focus our attention on the bifurcation to the antiphase phase synchronized state. It can be seen that, when ϵ increases from zero, $\langle s\rangle$ and $\sigma(s)$ decrease gradually. This shows the trend to an in-phase synchronized state as the coupling becomes stronger. At about $\epsilon\simeq 0.045$, $\langle s\rangle$ suddenly jumps to a quite large value which corresponds to the switch to the antiphase branch. The relatively small value of $\sigma(s)$ implies that the degree of synchronization for the antiphase state is higher than in the former in-phase branch.

In Sec. II, we showed that in the antiphase synchronized state a single oscillator behaves like a phase-coherent one although its parameter a is in the funnel regime. Therefore, a

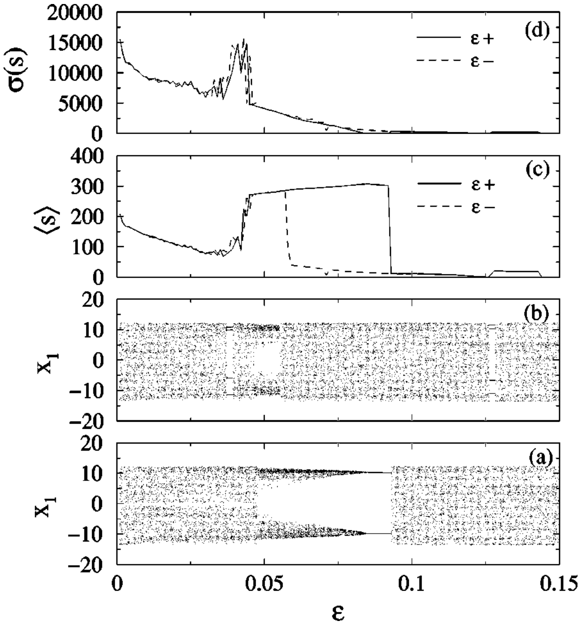

FIG. 2. Bifurcation diagram for the oscillator on site $i=1$ with (a) increasing and (b) decreasing coupling strength ϵ. Evolution of the variables $\langle s\rangle$ and $\sigma(s)$ with ϵ is shown in (c) and (d).

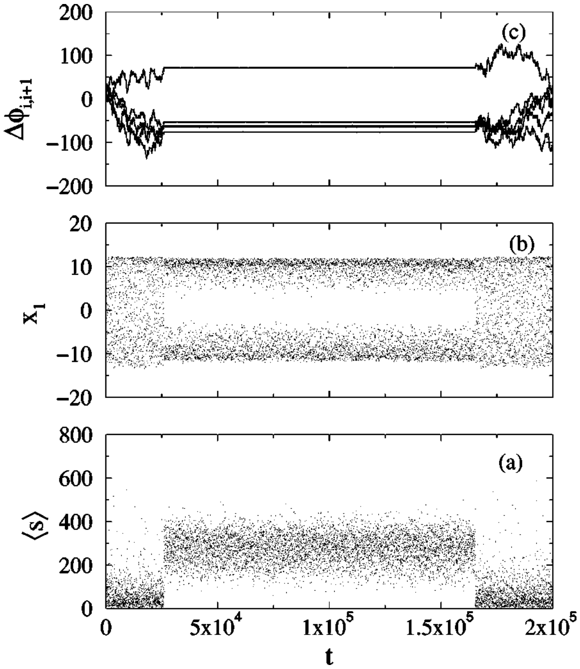

FIG. 3. Temporal evolution of (a) the average distance $\langle s\rangle$, (b) the variable x_1 for the oscillator on the site $i=1$, and (c) phase differences $\Delta\phi_{i,i+1}\equiv\phi_{i+1}-\phi_i$. Here the coupling strength is $\epsilon=0.0445$.

bifurcation in the local dynamic is expected on increasing the coupling strength. In Fig. 2(a), the value of x_1 on the Poincaré intersection $y_1=0$ is plotted with increasing ϵ. At $\epsilon\simeq0.045$, the one-band chaos suddenly shrinks in size to a two-band one. This is coincident with the time for the sudden jump in $\langle s\rangle$. This confirms our conjecture that there is a coupling-induced bifurcation from a funnel attractor to a phase-coherent one accompanying the transition to the antiphase state.

A large peak in $\sigma(s)$ appears prior to the bifurcation to the antiphase state. This is caused by the intermittency accompanying the transition: As ϵ is slightly smaller than the critical value 0.045, the chain of oscillators erratically switchs between segments of in-phase states and antiphase states (see Fig. 3). In the in-phase segments, $\langle s\rangle$ takes a relatively small value, the local dynamic for a single oscillator is a one-band chaos, and phase differences $\phi_i-\phi_{i+1}$ between neighbor oscillators diffuse randomly. In the antiphase segments, $\langle s\rangle$ has a large nonzero value, the attractor for a single site is a two-band chaotic one, and the phase differences $\phi_i-\phi_{i+1}$ stay almost constant. With increasing coupling strength, antiphase segments appear more and more frequently, and their average length diverges at the transition to the antiphase state.

Now we study the Lyapunov exponent spectrum for the coupled oscillators on increasing the coupling strength ϵ. The 12 largest Lyapunov exponents are plotted in Fig. 4. For $\epsilon=0$, without coupling, six of them are positive and the other six are zero corresponding to the free phases for autonomous oscillators. For two coupled phase-coherent oscillators, the phase synchronization occurs at the moment when one of the zero Lyapunov exponents becomes negative [7]. This means that one of the phase variables is no longer free and is locked to another. If this picture is still correct for the coupled funnel oscillators studied here, we expect the antiphase phase synchronization to happen when five of the six zero-value Lyapunov exponents become negative with only one left. The synchronized state should be a hyperchaotic state with N positive Lyapunov exponents and one zero Lyapunov exponent. Numerical calculation shows that this is not the case: With increasing ϵ, some of the zero-value Lyapunov exponents became negative gradually and some of

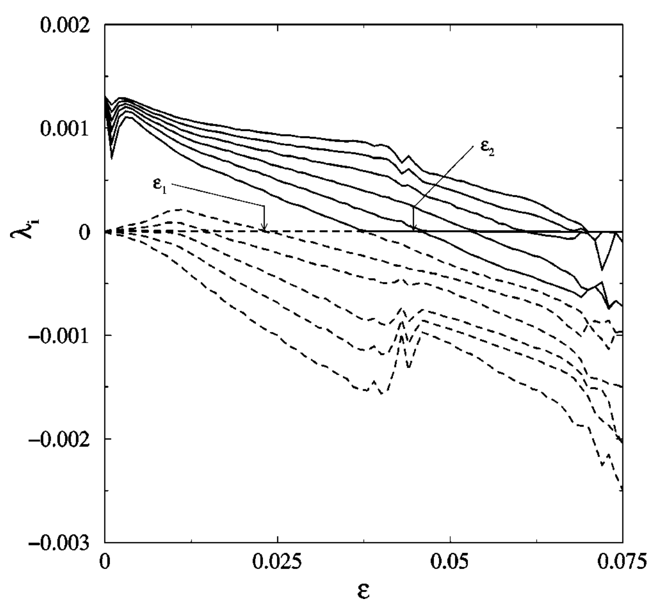

FIG. 4. Lyapunov exponent spectrum for coupled funnel oscillators with increasing coupling strength. Here $\epsilon_1\simeq0.023$ and $\epsilon_2\simeq0.045$.

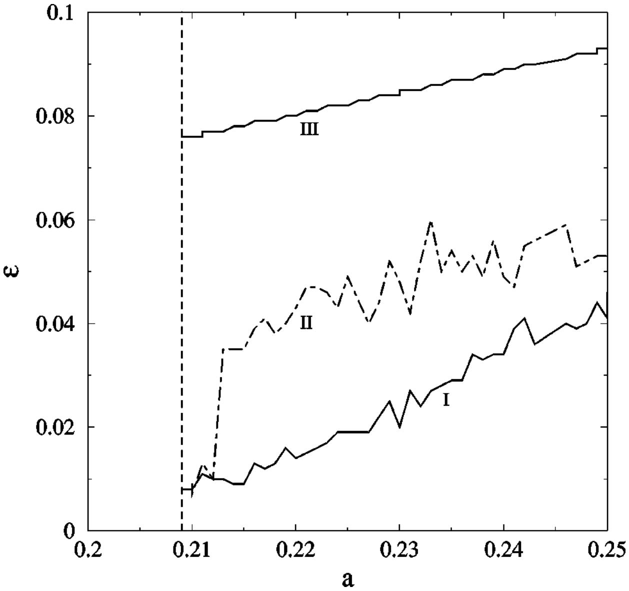

FIG. 5. Phase diagram for coupled Rössler oscillators. The dashed line marks the transition from a phase-coherent to a funnel attractor. The antiphase state exists in the region between lines I and III.

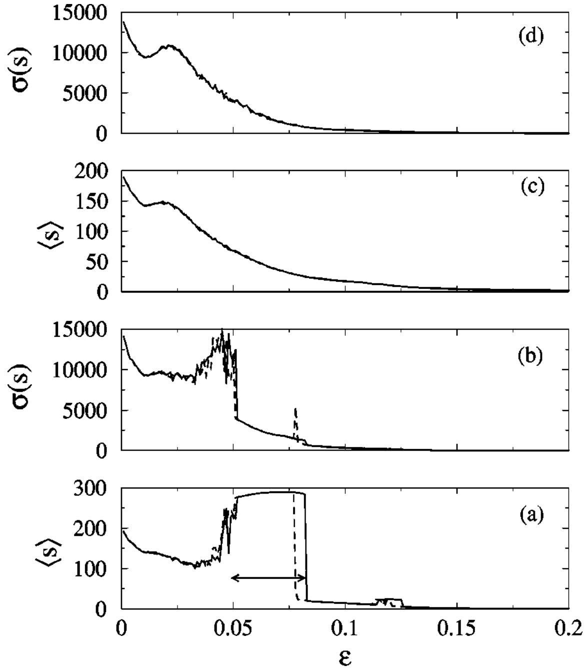

FIG. 6. Evolution of $\langle s\rangle$ and $\sigma(s)$ for nonidentical oscillators with the frequency difference $\Delta=0.03<\Delta_c$ in (a),(b) and with $\Delta=0.05>\Delta_c$ in (c),(d). Here the parameter $a=0.25$ and $N=6$.

them first become positive and then decrease to negative values. At $\epsilon\simeq0.023$, the system enters a state with six positive Lyapunov exponents and one zero-value Lyapunov exponent. This is quite far from the critical value 0.045 for the transition to the antiphase state. In fact, the antiphase phase synchronized chaotic state has only four positive Lyapunov exponents.

In Fig. 5, we plot the phase diagram on the parameter plane (a,ϵ) for our six-oscillator system. In the construction of this plot, for every value of the parameter a, we first increase ϵ slowly with the step $\delta\epsilon=0.001$, and two values of ϵ are recorded where $\langle s\rangle$ has sudden jumps. They are just the starting and ending points for the antiphase branch. Lines I and III in the plot are formed by them. Then the values of ϵ for sudden jumps in $\langle s\rangle$ with decreasing ϵ are also recorded. They are the end points for the in-phase branch continuing from the completely synchronized state. They form line II in the plot. The three lines disappear at $\epsilon\simeq0.021$ which is the critical value for the bifurcation from a phase-coherent to a funnel attractor in an uncoupled Rössler oscillator. This implies that the antiphase state exists only in a system of coupled funnel oscillators.

IV. EFFECT OF THE INHOMOGENEITY AND THE LATTICE SIZE

In this section, we would like to test the robustness of the antiphase state studied above. First, we check whether or not an inhomogeneity in natural frequencies ω_i will destroy this state. To this end, the natural frequencies are set to $\omega_i=1+\Delta\xi_i$ where ξ_i is a uniform random number in the interval $[-0.5,0.5]$. The evolution of $\langle s\rangle$ and $\sigma(s)$ for the case with $\Delta=0.03$ is shown in Fig. 6(a),(b). A similar structure as in Fig. 2 (for instance, sudden jumps and the hysteresis loop in $\langle s\rangle$) can be clearly seen here. To guide the eyes, the window of the phase-coherent state is marked by a line with arrows. When the inhomogeneity is increased to $\Delta=0.05$ [see Fig. 6(c),(d)], $\langle s\rangle$ changes smoothly with the coupling strength ϵ and neither sudden jumps nor a hysteresis loop appear. This means that the antiphase state survives under weak inhomogeneity while it disappears when the inhomogeneity is sufficiently strong. Detailed numerical simulations show that on increasing the inhomogeneity both the window of the phase-coherent state and the hysteresis loop gradually shrink their sizes. The critical value of the inhomogeneity beyond which no phase-coherent state appears is $\Delta_c\simeq0.04$.

Next we check whether the antiphase state still exists as we change the size of the chain. Numerical study shows that, when the size N is decreased, the antiphase state appears for the lattice size $N=4$ and 5. For $N\leqslant3$, it ceases to exist as shown in Fig. 7(a),(b). However, when N is increased, the antiphase state disappears for $N\geqslant N_c\simeq60$. The evolution of the variable $\langle s\rangle$ and $\sigma(s)$ for $N=70$ is shown in Fig. 7(c),(d). Sudden jumps and hysteresis loops are not observed here. The antiphase synchronized state, which is signaled by a relatively large value in the mean distance $\langle s\rangle$ and a small value in $\sigma(s)$, is not expected here. According to these results, we can say that the antiphase state exists only for intermediate values of the lattice size.

V. DISCUSSION AND CONCLUSIONS

In this paper we have studied the antiphase state for a chain of coupled funnel Rössler oscillators. Normally, the trajectory of a funnel attractor visits the origin frequently, which leads to 2π phase slips in the temporal evolution of the phase variable. If such funnel oscillators are coupled dif-

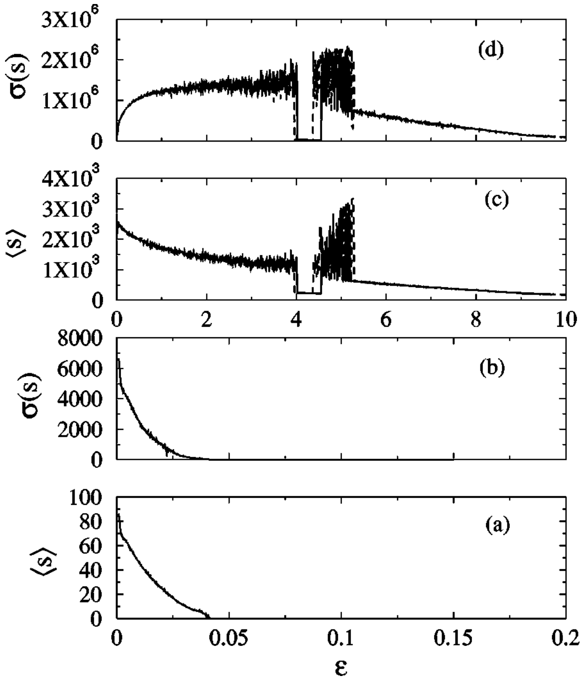

FIG. 7. Evolution of $\langle s\rangle$ and $\sigma(s)$ for coupled identical funnel oscillators with the chain size $N=3$ in (a),(b) and with $N=70$ in (c),(d). Here the parameter $a=0.25$.

fusively on a ring structure, an antiphase state appears when the coupling strength is strong enough. In such a state, phase differences between neighbor oscillators are of nearly constant value $2\pi/N$. The local dynamic on a single site changes from a funnel to a phase-coherent attractor. However, this antiphase state cannot be found in a system of coupled phase-coherent oscillators and is only expected for the funnel oscillator.

For the case of diffusively coupled small-amplitude oscillators near the Hopf bifurcation, Kuramoto argued that an antiphase state is possible when the effective coupling among phase variables becomes repulsive [13]. It is not expected that the same mechanism works in the case we studied here. Our reasons are as follows. (1) In our case, the amplitude plays an essential role since the funnel attractor is rendered phase coherent under coupling. For the case studied by Kuramato, the influence of coupling on the oscillator's amplitude is quite weak and using only phase variables is enough to describe the dynamics of the coupled oscillators. In other words, the case studied by Kuramoto is the weak-coupling regime while our case is far outside this regime. (2) The size effect observed here does not exist in Kuramoto's model. The antiphase state studied by Kuramoto survives at the thermodynamic limit.

The size-dependent effect stated in Sec. IV can be viewed in another way. We can consider only one unit in a system of N coupled oscillators while treating the impact from other units in the system as an environmental noise. This effective noise is expected to have a strength proportional to $1/\sqrt{N}$, where N is the size of the chain. In this sense, the appearance of a self-organized antiphase state for only an intermediate value of the chain size is a kind of stochastic resonance: The antiphase state appears at an optimal value of the effective noise which corresponds to an optimal value of the system size N. When N is quite large, the noise is too weak to trigger the antiphase state. However, for N quite small, the noise is strong in such a way that the ordered state is completely destroyed.

ACKNOWLEDGMENT

The author thanks A. Kenfack for careful reading of the paper.

[1] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, England, 1993).

[2] H. Fujisaka and T. Yamada, Prog. Theor. Phys. **69**, 32 (1983); A. S. Pikovsky, Z. Phys. B: Condens. Matter **55**, 149 (1984); L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).

[3] N. Platt, E. A. Spiegel, and C. Tresser, Phys. Rev. Lett. **70**, 279 (1993); H. L. Yang and E. J. Ding, Phys. Rev. E **50**, R3295 (1994); E. Ott and J. C. Sommerer, Phys. Lett. A **188**, 39 (1994).

[4] A. S. Pikovsky and P. Grassberger, J. Phys. A **24**, 4587 (1991); J. C. Alexander, J. A. Yorke, Z. You, and I. Kan, Int. J. Bifurcation Chaos Appl. Sci. Eng. **2**, 795 (1992); P. Ashwin, J. Buescu, and I. Stewart, Phys. Lett. A **193**, 126 (1994); Nonlinearity **9**, 703 (1996); Y. C. Lai, C. Grebogi, J. A. Yorke, and S. C. Venkataramani, Phys. Rev. Lett. **77**, 55 (1996).

[5] N. F. Rulkov, M. M. Suschik, L. S. Tsimring, and H. D. I. Abarbanel, Phys. Rev. E **51**, 980 (1995); U. Parlitz, L. Junge, W. Lauterborn, and L. Kocarev, *ibid.* **54**, 2115 (1996).

[6] V. Astakhov, M. Hasler, T. Kapitaniak, A. Shabunin, and V. Anishchenko, Phys. Rev. E **58**, 5620 (1998); R. L. Viana and C. Grebogi, *ibid.* **62**, 462 (2000); Meng Zhan and Gang Hu, *ibid.* **62**, 375 (2000).

[7] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996).

[8] A. Pikovsky, G. Osipov, M. Rosenblum, M. Zaks, and J. Kurths, Phys. Rev. Lett. **79**, 47 (1997).

[9] A. Pikovsky, M. Rosenblum, M. Zaks, and J. Kurths, in *Handbook of Chaos Control*, edited by H. G. Schuster (Wiley-VCH, Weinheim, 1999), pp. 252–273, and references therein.

[10] G. Hu, Y. Zhang, H. A. Cerdeira, and S. Chen, Phys. Rev. Lett. **85**, 3377 (2000).

[11] For nonidentical units see G. Osipov, A. S. Pikovsky, M. G. Rosenblum, and J. Kurths, Phys. Rev. E **55**, 2353 (1997); for identical units see L. Brunnet, H. Chate, and P. Manneville, Physica D **78**, 141 (1994); A. Goryachev and R. Kapral, Phys. Rev. Lett. **76**, 1619 (1996).

[12] See, for example, G. Hu, J. Z. Yang, W. Q. Ma, and J. H. Xiao, Phys. Rev. Lett. **81**, 5314 (1998).

[13] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, Berlin, 1984).